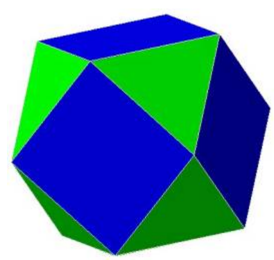
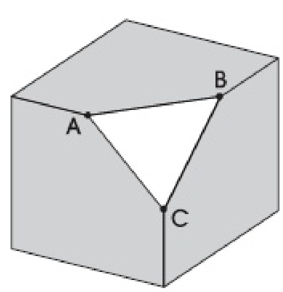
The Invariance Principle

What do five fingers and five sheep have in common? Let's say we line-up the sheep from left to right in some order. If we begin with an outstretched hand, fist unclenched, and then proceed to pull-in the fingers one-by-one, in order from left to right, one finger for every sheep, we end up with a clenched fist precisely as we run out of sheep. We have just *counted* the sheep, and found that there are as many of them as there are fingers on one of our hands. Indeed, if we repeat this process with a different ordering of the sheep from left to right, we end up with the same result. It doesn't matter how we count them - we always find that there are as many sheep as there are fingers on one hand. Their *number* remains the same. We say that the number of sheep is *invariant* under the operation of counting.

Here's another example. Can you find an interesting property shared by a cube and the solid shown below?



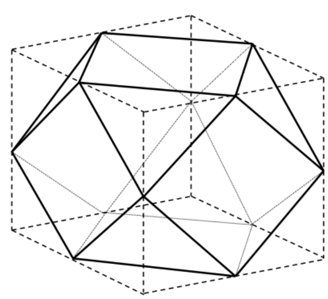
Suppose we begin with the cube. Let's count the number of vertices, the number of edges, and the number of faces. There are 8 vertices, 12 edges, and 6 faces. We'll call these numbers V, E and F. Next, let's compute the number $\chi := V - E + F$. This number is called the *Euler characteristic* of the solid. For our cube, we find that $\chi = 2$. Now, slice-off one of the cube's eight vertices in such a manner that the plane of the slice bisects each of the three edges meeting at that vertex.



This creates a new triangular face in place of the vertex. (Triangle ABC in the picture above.) Note that the slicing operation gets rid of the original vertex, but introduces a brand new face, which itself has 3 edges and 3 vertices.

So the first time we slice-off a vertex of the cube, we increase F by 1, E by 3 and V by 2. But this means that there is no net change in χ at all. Let's continue with our slicing operation and excise a second vertex. If we were to slice-off one of the four vertices that were not connected to the first vertex by an edge, we'd see that the same thing happens – χ remains unchanged. To make things a little more interesting, we'll instead slice-off one of the three vertices that was connected to the first vertex by an edge. This time, we again increase F by 1, but we only increase E and V by 2 each. This is because the newly created triangular face, in addition to introducing three edges of its own, now also eliminates one of the previous edges – the one that connected the two original vertices of the cube. And, it also shares one of its vertices with a vertex of the previous triangular face. Nevertheless, it's again clear that χ remains unchanged.

We leave it to you to convince yourself that no matter what order we choose to slice-off all eight vertices in, the Euler characteristic remains unchanged after each slicing operation. Indeed, it's easy to see that when we are done slicing-off all eight vertices in the prescribed manner, we end up with precisely the solid shown in the first picture.



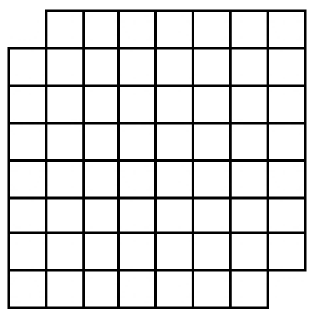
And this solid has 12 vertices, 24 edges, and 14 faces. So its Euler characteristic is – still 2! In other words, the Euler characteristic is invariant under the operation of slicing-off vertices.

In general, given a dynamical system $f : X \to X$, any properties that are unchanged along orbits are called *invariants* of the system. Knowing these invariants helps us distinguish orbits. If two points have different values of a particular invariant, they must be on different orbits. In other words, we cannot get from one to the other by applying the operation f. We will call this *the invariance principle.*

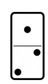
More concretely, let's say we are studying a universe of objects in which two objects are regarded as essentially the same if one can be converted into the other by performing a specified reversible operation. Because two objects in this universe that look different may nevertheless turn out to be essentially the same, the only way to be sure that they really are different is to find an invariant of the system which takes different values for those two objects.

The quest for suitable invariants for different kinds of dynamical systems is ubiquitous in mathematics. The following questions will give you a taste of what this quest feels like.

1. Shown below is an 8×8 grid of squares with a pair of diagonally opposite corners surgically removed.

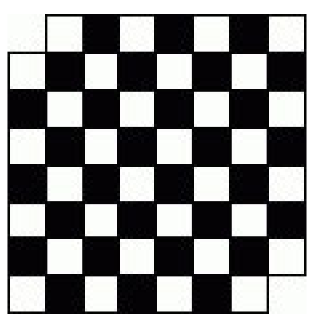


You are given 31 dominoes, each 2 squares long and 1 square wide.



Can you cover this post-surgery grid by placing the dominos on it, so that each domino is positioned either horizontally or vertically, and exactly covers 2 squares on the underlying grid?

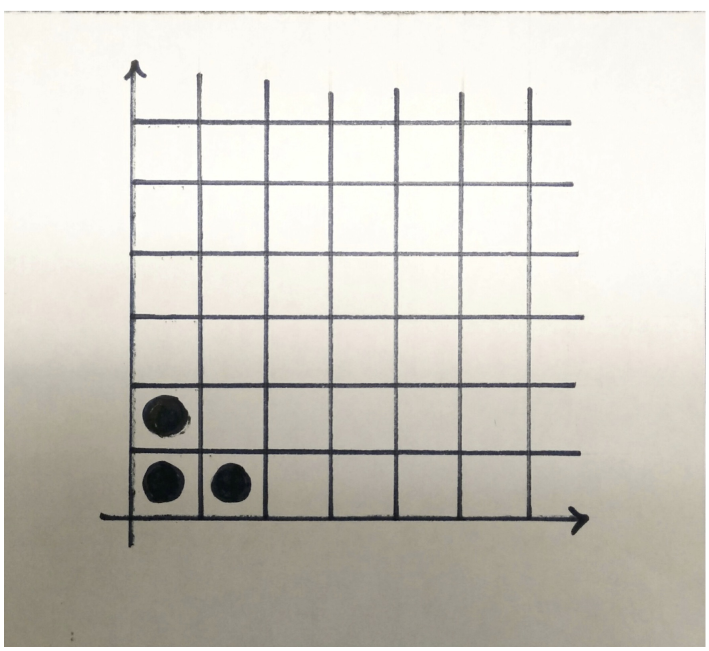
Hint:



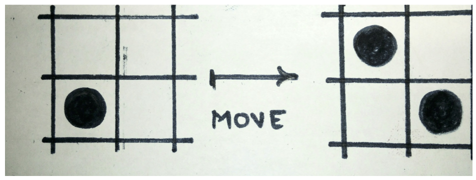
How about now?

Solution: A grid of squares is pretty special amongst polygonal tilings of the plane - no three tiles are such that any two among them share an edge. This means that we can color the tiles using just two colors and have no two adjacent tiles share the same color. Consider the chessboard pattern shown in the hint. If we were to place a domino on the grid as prescribed, clearly it would cover exactly one square of each color. In other words, the difference between the number of black and white squares is invariant under the act of placing dominos on the grid. Since covering the board exactly and fully would bring that difference down from two to zero, it cannot be accomplished by placing dominos as prescribed.

2. Imagine overlaying a regular pattern of squares on the first quadrant of the Cartesian plane, so that we get a grid that stretches infinitely in the upward and rightward directions, and has precisely one corner – the bottom-left one. Now place three coins on the grid – one in each of the three squares near the corner – as shown below.

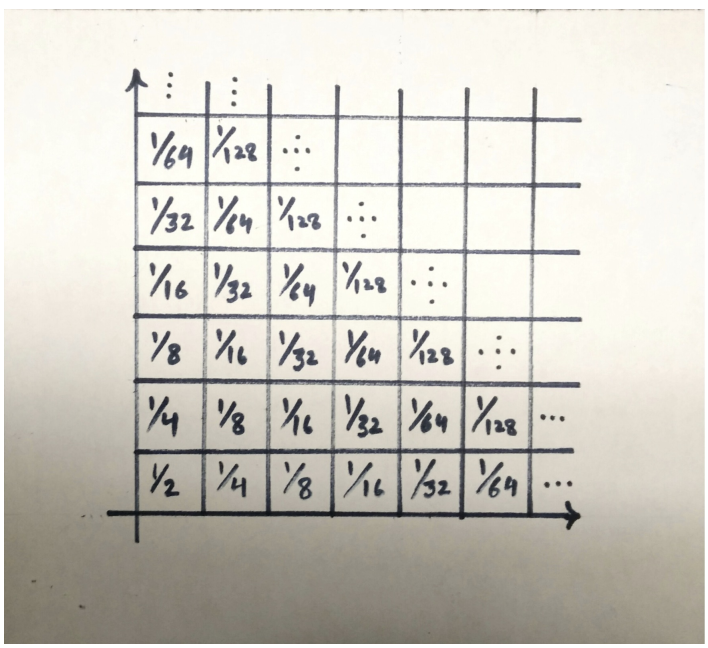


We'll play a game. You are the only player, and there is just one legal move - if a square occupied by a coin has an empty square right above it as well as an empty square beside it on the right, you can remove the coin in that square and replace it with two coins, one in each of the two empty adjacent squares mentioned above.



The goal is to help all three initial coins escape from the squares they were imprisoned in. In other words, you want to execute a finite sequence of moves at the end of which the three squares near the corner are empty. Can you do it?

Hint:



How about now?

Solution: Again, the hint tells us how to construct an invariant for this system. Define the total weight of the system as the number obtained by adding up all the numbers in occupied squares. It is clear that this is invariant under any move. Now, the initial weight of the system is

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1.$$

Vacating the three squares initially occupied will therefore require shifting this weight into other squares. However, the maximum possible weight of the system cannot exceed the weight of the grid when all squares are occupied. Knowing how to sum geometric series, we see that this upper bound is 2. In other words, vacating the initial three squares requires filling up all of the infinitely many other squares. This is impossible to achieve in finitely many moves.