Introduction to Propositional Logic

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1. Propositional Logic

Goals:
The characteristic thinking of a mathematician is **deductive reasoning**, in which one uses logic to draw conclusions based on statements accepted as true. Our goal in this lesson is to provide a working knowledge of the basics of logic and the idea of proofs, which are fundamental to deductive reasoning. This type of thinking is useful in many areas in addition to math. For example, thought processes used to construct an algorithm for a computer program are much like those used to develop the proof of a theorem.

Definition 1.1. A **proposition** is a sentence that is either true or false, but NOT both.

A proposition has exactly one truth value: true, which we denote by T, or false, which we denote by F.

Examples 1.2. (1) 2 is an even number. T

(2) $1 + 1 = 5$. F

(3) You will earn 1 million dollars before you die.

This is a proposition since it can be determined and can not be both T and F. Currently its truth value is unknown.

(4) Martin Luther King Jr. had two eggs for breakfast on his tenth birthday.

We may never know the truth value, but it has one and only one so it is a proposition.

(5) What did you say?

NOT a proposition since it has no truth value.

(6) Next year Hope Academy will have exactly 100 students.

This is a proposition since it can be determined and can not be both T and F. Currently its truth value is unknown.

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(7) $5 + x = 9$.

NOT a proposition since the truth value changes depending on the value of x.

Students do **Understanding Propositions** worksheet.

Definition 1.3. A **propositional variable** is a variable used to represent a proposition.

It is standard to use P, Q, R, S, or p, q, r, s as propositional variables. Many mathematical statements are constructed by combining one or more propositions.

Definition 1.4. A **compound proposition** is a proposition constructed by combining propositions using logical operators. These logical operators are also called **connectives**.

Types of Compound Propositions: Let P and Q be propositions.

1. **Conjunction:** the conjunction of P and Q, denoted $P \wedge Q$, is the proposition "P and Q." $P \wedge Q$ is true exactly when BOTH P and Q are true.

2. **Disjunction:** the disjunction of P and Q, denoted $P \vee Q$, is the proposition "P or Q." $P \vee Q$ is true exactly when at least one of P or Q is true.

3. **Negation:** the negation of P, denoted $\neg P$ (or $\overline{P}$, or $\sim P$), is the proposition "not P." $\neg P$ is true exactly when P is false.

Examples 1.5. Examples of Compound Propositions. Determine the truth values.

(1) If P is "$1 \neq 3$" and Q is "7 is odd", then $P \wedge Q$ is T, $P \vee Q$ is T, and $\neg Q$ is F.

(2) It is not the case that $3 > 4$. T

(3) President Obama will live to be 85 years old and 4 is a prime number. F

(4) It is not the case that 10 is divisible by 2. F

(5) $2 < 3$ or chickens have lips. T

(6) $\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$ and the University of Houston has only one building. F

(7) Venus is smaller than Earth or $1 + 4 = 5$. T

(8) $6 < 7$ and $7 < 8$. T

Definition 1.6. A **truth table** is a table displaying the truth values of propositions.

Examples 1.7. (1) Truth table for $\neg P$

P	$\neg P$
T	F
F	T

(2) Truth table for the conjunction and disjunction.

P	Q	$P \wedge Q$	$P \vee Q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

We often encounter compound propositions with more than 2 simple components. The propositional form $(P \wedge Q) \vee \neg R$ has 3 simple components, P, Q, and R. It follows that there are $2^3 = 8$ possible combinations of truth values. The two main components are $P \wedge Q$ and $\neg R$. We make truth tables for these and combine them by using the truth table for $\vee$.

P	Q	R	$P \wedge Q$	$\neg R$	$(P \wedge Q) \vee \neg R$
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	F	T	T
F	T	T	F	F	F
F	T	F	F	T	T
F	F	T	F	F	F
F	F	F	F	T	T

Definition 1.8. A system of propositions is **consistent** when it is possible for all statements to be true under the same conditions.

P	Q	R	$P \wedge Q$	$\neg R$	$(P \wedge Q) \vee \neg R$
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	F	T	T
F	T	T	F	F	F
F	T	F	F	T	T
F	F	T	F	F	F
F	F	F	F	T	T

This system is consistent because of the second line of the truth table.

Why do we care about consistent systems? In computer science this is a way to tell if it is possible to design a computer program that can do several tasks.