

PATTERNS AND GAMES

“Last One Standing”

Four people are sitting on chairs in a line all facing the same direction, so each person can see all the people in front of them and none of the people in back of them. We start with everyone sitting, and each turn we can make a move and have one person change position (either stand up or sit down) subject to the following rules:

Rule 1: The front person is allowed to change position (stand or sit) on any turn.

Rule 2: All other people can change position (stand or sit) only if the person in front of them is standing and all other people in front of them are sitting.

Our goal is to have the back person standing and all the other people sitting. We call this a *winning position*.

QUESTION 1: What is the fewest number of moves we need to put 4 people who are all sitting into a winning position?

QUESTION 2: How many moves would it take if we started with 10 people sitting in a line? How many moves would it take if we started with 15 people sitting in a line? How many moves would it take if we started with 20 people sitting in a line?

QUESTION 3: If we did one move per second (so one person sits or one person stands each second), how long would it take to move to a winning position if we start with 4 people? If we start with 10 people? If we start with 15 people? If we start with 20 people?

Answers: For question 1 we can just play the game and find it takes 15 moves. For Questions 2 and 3, it helps to consider the game with 1 person, 2 people, 3 people, etc. and look for a pattern.

Number of People	Number of Moves Needed	Time Required
1	1	1 second
2	3	3 seconds
3	7	7 seconds
4	15	15 seconds
5	31	31 seconds
\vdots	\vdots	\vdots
n people	$2^n - 1$	$2^n - 1$ seconds
\vdots	\vdots	\vdots
10	$2^{10} - 1 = 1023$	17 minutes
15	$2^{15} - 1 = 32,767$	9 hours
20	$2^{20} - 1 = 1,048,575$	12 days

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“Nim”

Two players play a game. There are two piles that each have 5 stones, and the players take turns.

Rule 1: On each person’s turn they must take away 1 or more stones. However, all stones taken must be from the same pile.

Rule 2: The person who takes the last stone wins.

QUESTION 1: Break into groups of two and play this game a few times. Can you figure out a strategy to win? Would you rather go first or go second in this game?

Answer: We see that you can always win if you go second. On your move simply make sure that you take from the pile with the most stones and you remove exactly the number needed to make the piles have an equal number. If you do this, eventually, you opponent will have to remove all stones from one pile, and then you can take all stones (and hence the last stone) from the remaining pile.

QUESTION 2: What happens if we change the number of stones so there are 6 in each pile? So there are 7 in each pile?

Answer: Our strategy from Question 1 works for any number of stones, as long as the two piles have an equal number.

QUESTION 3: What if we changed the game so we had two piles that have an unequal number of stones? Would this change your strategy? Would you want to go first or second in this game?

Answer: We see that the same strategy will work as long as you get to go first. If you go first, simply make sure that you take from the pile with the most stones and you remove exactly the number needed to make the piles have an equal number. Then it is the opponents turn and you’re in the same situation as before.

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“Twenty-One”

Two players play a game. We start by writing the number 0 on a piece of paper.

Rule 1: On each person’s turn, they may add 1, 2, or 3 to the existing number. A player is not allowed to pass or to add zero.

Rule 2: The first person to reach 21 (or more) wins.

QUESTION 1: Break into groups of two and play this game a few times. Can you figure out a strategy to win? Would you rather go first or go second in this game?

Answer: If you can get the total to 17, that is a good number, because whatever your opponent does on the next move allows you to get to 21 on the following move.

Likewise, 13 is a good number, because whatever your opponent does on the next move allows you to get to 17 on the following move.

9 is a good number, because whatever your opponent does on the next move allows you to get to 13 on the following move.

5 is a good number, because whatever your opponent does on the next move allows you to get to 9 on the following move.

1 is a good number, because whatever your opponent does on the next move allows you to get to 5 on the following move.

So the strategy is to go first, and on your first move add 1 to 0 to make the total 1. Then after each of your opponent’s moves, add enough to get to a good number: 5, 9, 13, 17, or 21. When you get to 21 you win.

QUESTION 2: What happens if we change the winning number to 22? Would this change your strategy? Would you want to go first or second in this game?

Answer: The good numbers are now 22, 18, 14, 10, 6, 2. You still want to go first, but now your first move is to 2 to 0 to make the total 2. Then after each of your opponent’s moves, add enough to get to a good number: 6, 10, 14, 18, or 22. When you get to 22 you win.

QUESTION 3: What happens if we change the winning number to 20? Would this change your strategy? Would you want to go first or second in this game?

Answer: The good numbers are now 20, 16, 12, 8, 4, and 0. Now, when you start the game, 0 is already a good number. So you want to go second. Then whatever your opponent does you want to add enough to get to a good number: 4, 8, 12, 16, or 20. When you get to 20 you win.

QUESTION 4: What if we kept the winning number at 21, but we changed the game so you are allowed to add 1, 2, 3, or 4 on any turn? Would this change your strategy? Would you want to go first or second in this game?

Answer: The good numbers are now 21, 16, 11, 6, and 1. So the strategy is to go first, and on your first move add 1 to 0 to make the total 1. Then after each of your opponent's moves, add enough to get to a good number: 6, 11, 16, or 21. When you get to 21 you win.